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PROFESSOR CAYLEY, President, in the Chair.

Rev. Samuel Jenkins Johnson, Upton Helions, near Crediton;
George Walter Roberts, Esq., Gomersal, near Leeds;
Alfred Herrtage, Esq., University College,

were balloted for and duly elected Fellows of the Society.

On a Pair of Differential Equations in the Lunar Theory.

By Prof. Cayley.

I consider the differential equations

$$\frac{d}{dt}\frac{d\varrho}{dt} - \varrho\left(\frac{dv}{dt}\right)^2 + \frac{1}{\varrho^2} = k m^2 \varrho\left\{\frac{1}{2} + \frac{3}{2}\cos(2v - 2mt)\right\},$$

$$\frac{d}{dt}\left(\varrho^2\frac{dv}{dt}\right) \qquad = j m^2 \varrho^2\left\{\quad - \frac{3}{2}\sin(2v - 2mt)\right\},$$

which when $j = k = 1$ give the following equations in the lunar theory ($D = t - mt$):

$$\frac{1}{\varrho} = 1 + \frac{1}{6}m^2 - \frac{179}{288}m^4 - \frac{97}{48}m^5 - \frac{757}{162}m^6 - \frac{4039}{432}m^7 - \frac{34751189}{1990656}m^8 - \frac{155067635}{4976640}m^9$$

$$+ \cos 2D \left[m^2 + \frac{19}{6}m^3 + \frac{131}{18}m^4 + \frac{383}{27}m^5 + \frac{510565}{20736}m^6 + \frac{23140781}{622080}m^7 \right.$$
$$\left. + \frac{355021217}{9331200}m^8 + \frac{27888590059}{34992000}m^9\right]$$

$$+ \cos 4D \left[\frac{7}{8}m^4 + \frac{2737}{480}m^5 + \frac{162869}{7200}m^6 + \frac{7554833}{108000}m^7 + \frac{2389416723}{12960000}m^8 \right.$$
$$\left. + \frac{233523012528 3}{5443200000}m^9\right]$$

$$+ \cos 6D \left[\frac{219}{256}m^6 + \frac{151339}{17920}m^7 + \frac{29887443}{627200}m^8 + \frac{98978623957}{444528000}m^9\right],$$

$$+ \cos 8D \left[\frac{2701}{3072}m^8 + \frac{70033633}{6021120}m^9\right],$$

or as far as m^7,

$$\varrho = 1 - \frac{1}{6}m^2 + \frac{331}{288}m^4 + \frac{83}{16}m^5 + \frac{42775}{2592}m^6 + \frac{4787}{108}m^7$$

$$+ \cos 2D \left[-m^2 - \frac{19}{6}m^3 - \frac{125}{18}m^4 - \frac{709}{54}m^5 - \frac{485173}{20736}m^6 - \frac{24487949}{622080}m^7\right]$$

$$+ \cos 4D \left[-\frac{3}{8}m^4 - \frac{1217}{480}m^5 - \frac{74069}{7200}m^6 - \frac{1749779}{54000}m^7\right]$$

$$+ \cos 6D \left[-\frac{59}{256}m^6 - \frac{126193}{53760}m^7\right],$$

($\frac{1}{\varrho}$ is given by M. Delaunay only as far as m^5, the additional terms of $\frac{1}{\varrho}$ and expression for ϱ were kindly communicated to me by Prof. Adams); and

$$v = t$$
$$+ \sin 2D \left(\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{893}{72}m^4 + \frac{2855}{108}m^5 + \frac{8304449}{165888}m^6 \right.$$
$$\left. + \frac{102859909}{1244160}m^7 + \frac{7596606727}{74649600}m^8 - \frac{8051418161}{111974400}m^9\right)$$

$$+ \sin 4D \left(\frac{201}{256}m^4 + \frac{649}{120}m^5 + \frac{647623}{28800}m^6 + \frac{31363361}{432000}m^7 + \frac{123030377303}{414720000}m^8\right)$$

$$+ \sin 6\,D \left(\frac{3715}{6144} m^6 + \frac{664571}{107520} m^7\right)$$

(Delaunay, t. xi. pp. 815, 836, 845.)

To integrate the original equations write

$$\begin{aligned} \rho &= 1 + \rho_1 + \rho_2 + \ldots \\ v &= t + v_1 + v_2 + \ldots \end{aligned}$$

where the suffixes indicate the degrees in the co-efficients k, j conjointly: the equations for ρ_n, v_n take the form

$$\frac{d}{dt}\frac{d\rho_n}{dt} - 3\rho_n - 2\frac{dv_n}{dt} + V_n = Q_n,$$

$$\frac{d}{dt}\left(\frac{dv_n}{dt} + 2\rho_n + U_n\right) \qquad = P_n,$$

where V_n, U_n, P_n, Q_n do not contain ρ_n or v_n. From the second equation we have

$$\frac{dv_n}{dt} + 2\rho_n + U_n = \Omega_n + \int P_n\,dt,$$

where Ω_n is a constant of integration, the integral $\int P_n\,dt$ containing only periodic terms; and then adding twice this to the first equation we have

$$\frac{d}{dt}\frac{d\rho_n}{dt} + \rho_n + V_n + 2U_n = 2\Omega_n + Q_n + 2\int P_n\,dt$$

which determines ρ_n; and substituting its value in the other equation we have $\frac{dv_n}{dt}$, and thence v_n; the constant Ω_n is determined so that $\frac{dv_n}{dt}$ may contain no constant term. We have

$$V_1 = 0,$$

$$V_2 = -\left(\frac{dv_1}{dt}\right)^2 - 2\rho_1\frac{dv_1}{dt} + 3\rho_1^2,$$

$$V_3 = -2\frac{dv_1}{dt}\frac{dv_2}{dt} - 2\rho_1\frac{dv_2}{dt} - \rho_1\left(\frac{dv_1}{dt}\right)^2$$
$$- 2\rho_2\frac{dv_2}{dt} + 6\rho_1\rho_2 - 4\rho_1^3,$$

&c.

$$U_1 = 0,$$

$$U_2 = 2\rho_1\frac{dv_1}{dt} + \rho_1^2,$$

$$U_3 = 2\rho_1\frac{dv_2}{dt} + (2\rho_2 + \rho_1^2)\frac{dv_1}{dt} + 2\rho_1\rho_2,$$

&c.

$$Q_1 = k\,m^2\left(\frac{1}{2} + \frac{3}{2}\cos 2\,D\right),$$

$$Q_2 = k\,m^2\left\{3\,v_1 \sin 2\,D + \varrho_1\left(\frac{1}{2} + \frac{3}{2}\cos 2\,D\right)\right\},$$

$$Q_3 = k\,m^2\left\{-\,3\,v_2 \sin 2\,D - 3\,v_1{}^2 \cos 2\,D + \varrho_1 v_1 \cdot 3 \sin 2\,D + \varrho_2\left(\frac{1}{2} + \frac{3}{2}\cos 2\,D\right)\right\},$$

&c.

$$P_1 = j\,m^2\left(-\frac{3}{2}\sin 2\,D\right),$$

$$P_2 = j\,m^2\,(-\,3\,v_1 \cos D - 3\,\varrho_1 \sin 2\,D)$$

$$P_3 = j\,m^2\left\{-\,3\,v_2 \cos 2\,D + 3\,v_1{}^2 \sin 2\,D, - 6\,\varrho_1 v_1 \cos 2\,D + (2\,\varrho_2 + \varrho_1{}^2)\cdot -\frac{3}{2}\sin 2\,D\right\},$$

&c.

In particular attending to the values of P_1, Q_1 the equations for ϱ_1, v_1 are in their original form

$$\frac{d}{dt}\frac{d\,\varrho_1}{dt} - 3\,\varrho_1 + 2\,\frac{d\,v_1}{dt} = k\,m^2\left(\frac{1}{2} + \frac{3}{2}\cos 2\,D\right),$$

$$\frac{d}{dt}\left(\frac{d\,v_1}{dt} + 2\,\varrho_1\right) = j\,m^2\left(-\frac{3}{2}\sin 2\,D\right),$$

whence in the transformed form they are

$$\frac{d\,v_1}{dt} + 2\,\varrho_1 = \Omega_1 + \frac{3\,j\,m^2}{4\,(1 - m)}\cos 2\,D,$$

and

$$\frac{d^2\varrho_1}{dt^2} + \varrho_1 = 2\,\Omega_1 + k\,m^2\left(\frac{1}{2} + \frac{3}{2}\cos 2\,D\right) + \frac{\frac{3}{2}\,j\,m^2}{1 - m}\cos 2\,D.$$

Thus the constant term of ϱ_1 is $2\,\Omega_1 + \frac{1}{2}k\,m^2$, giving in $\frac{d\,v_1}{dt}$ a constant term $-\,3\,\Omega_1 - k\,m^2$; this must vanish, or we have $\Omega_1 = -\frac{1}{3}k\,m^2$; and the equations thus become

$$\frac{d\,v_1}{dt} + 2\,\varrho_1 = -\frac{1}{3}k\,m^2 + \frac{3\,j\,m^2}{4\,(1 - m)}\cos 2\,D,$$

$$\frac{d^2\varrho_1}{dt^2} + \varrho_1 = -\frac{1}{6}k\,m^2 + \left(\frac{3}{2}k\,m^2 + \frac{\frac{3}{2}\,j\,m^2}{1-m}\right)\cos 2\,D,$$

and then completing the integration

$$\varrho_1 = -\frac{1}{6}k\,m^2 + \left\{\frac{-\frac{3}{2}k\,m^2}{3 - 8\,m + 4\,m^2} + \frac{-\frac{3}{2}j\,m^2}{(1 - m)\,(3 - 8\,m + 4\,m^2)}\right\}\cos 2\,D,$$

$$v_1 = \left\{\frac{\frac{3}{2}k\,m^2}{(1-m)(3-8\,m+4\,m^2)} + \frac{\frac{3}{8}j\,m^2\,(7-8\,m+4\,m^2)}{(1-m)^2\,(3-8m+4\,m^2)}\right\}\sin 2\,D.$$

which are the accurate values of ϱ_1 and v_1.

Expanding as far as m^6 we have

$$\varrho_1 = k\left(-\frac{1}{6}m^2\right) + \cos 2\mathrm{D}\left\{ k\left(-\frac{1}{2}m^2 - \frac{4}{3}m^3 - \frac{26}{9}m^4 - \frac{160}{27}m^5 - \frac{968}{81}m^6\right)\right.$$

$$\left. + j\left(-\frac{1}{2}m^2 - \frac{11}{6}m^3 - \frac{85}{18}m^4 - \frac{575}{54}m^5 - \frac{3661}{162}m^6\right)\right\}$$

which for $j = k$ is $$= k\left(- m^2 - \frac{19}{6}m^3 - \frac{137}{18}m^4 - \frac{895}{54}m^5 - \frac{5597}{162}m^6\right)$$

and

$$v_1 = \sin 2\mathrm{D}\left\{ k\left(\frac{1}{2}m^2 + \frac{11}{6}m^3 + \frac{85}{18}m^4 + \frac{575}{54}m^5 + \frac{3661}{162}m^6\right)\right.$$

$$\left. + j\left(\frac{7}{8}m^2 + \frac{37}{12}m^3 + \frac{589}{72}m^4 + \frac{1037}{54}m^5 + \frac{27331}{648}m^6\right)\right\}$$

which for $j = k$ is $$= k\left(\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{929}{72}m^4 + \frac{896}{27}m^5 + \frac{41975}{648}m^6\right).$$

I have, not in general, but for the value $j = k$, calculated ϱ_2 and v_2 as far as m^6: I have not made the calculation for ϱ_3 and v_3, but their values may be deduced from the foregoing values of ϱ, v; the final expressions (when $j = k$) of ϱ, $= 1 + \varrho_1 + \varrho_2 + \varrho_3 + \ldots$ and v, $= t + v_1 + v_2 + v_3 \ldots$ are

$$\begin{aligned}
\varrho = 1 & \\
& + k\left(-\frac{1}{6}m^2\right) \\
& + k^2\left(\frac{331}{288}m^4 + \frac{83}{16}m^5 + \frac{5113}{288}m^6\right) \\
& + k^3\left(-\frac{1621}{1296}m^6\right) \\
+ \cos 2\mathrm{D}\Big\{ & k\left(- m^2 - \frac{19}{6}m^3 - \frac{137}{18}m^4 - \frac{895}{54}m^5 - \frac{5597}{162}m^6\right) \\
& + k^2\left(\frac{2}{3}m^4 + \frac{31}{9}m^5 + \frac{329}{27}m^6\right) \\
& + k^3\left(-\frac{2381}{2304}m^6\right)\Big\} \\
+ \cos 4\mathrm{D}\Big\{ & k^2\left(-\frac{3}{8}m^4 - \frac{1217}{480}m^5 - \frac{76589}{7200}m^6\right)
\end{aligned}$$

$$+k^3\Big(\qquad\qquad\qquad\qquad\qquad\qquad + \frac{7}{20}\ m^6\Big)\Big\}$$

$$+\cos 6\,\mathrm{D}\ \Big\{\ k^3\Big(\qquad\qquad\qquad\qquad\qquad\qquad - \frac{59}{256}\ m^6\Big)\Big\}.$$

and

$$v = t$$

$$+\sin 2\,\mathrm{D}\ \Big\{\ k\Big(\frac{11}{8}m^2 + \frac{59}{12}m^3 + \frac{929}{72}m^4 + \frac{896}{27}m^5 + \frac{41975}{048}\ m^6\Big)$$

$$+k^2\Big(\qquad\qquad - \frac{1}{2}\ m^4 - \frac{41}{12}\ m^5 - \frac{43}{3}\ m^6\Big)$$

$$+k^3\Big(\qquad\qquad\qquad\qquad\qquad\qquad - \frac{783}{2048}\ m^6\Big)\Big\}$$

$$+\sin 4\,\mathrm{D}\ \Big\{\ k^2\Big(\qquad\qquad \frac{201}{256}m^4 + \frac{649}{120}m^5 + \frac{665263}{28800}m^6\Big)$$

$$+k^3\Big(\qquad\qquad\qquad\qquad\qquad\qquad - \frac{49}{80}\ m^6\Big)\Big\}$$

$$+\sin 6\,\mathrm{D}\Big\{\ k^3\Big(\qquad\qquad\qquad\qquad\qquad\qquad + \frac{3715}{6144}\ m^6\Big)\Big\};$$

which for $k = 1$ agree with the foregoing formulæ (verifying them as far as m^5); the present formulæ exhibit the manner in which the expressions depend on the several powers of the disturbing force.

On the Variations of the Position of the Orbit in the Planetary Theory. By Prof. Cayley.

It has always appeared to me that in the Planetary Theory, more especially when the method of the variation of the elements is made use of, there is a difficulty as to the proper mode of dealing with the inclinations and longitudes of the nodes, hindering the ulterior development of the theory. Considering the case of two planets m, m', and referring their orbits to any fixed plane and fixed origin of longitudes therein, let θ, θ' be the longitudes of the nodes, ϕ, ϕ' the inclinations ($p = \tan\phi\sin\theta$, $q = \tan\phi\cos\theta$, &c., as usual); then the disturbing functions for m, m' respectively are developed, not explicitly in terms of ϕ, ϕ', θ, θ', but in terms of Φ, the mutual inclination of the two orbits, and of Θ, Θ' the longitudes in the two orbits respectively of the mutual node of the two orbits; Φ and Θ, Θ' being functions (and complicated